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THE EFFECT OF EARTH'S OBLATENESS ON THE CALCULATION OF THE IMPACT POINT OF BALLISTIC MISSILES

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THE EFFECT OF EARTH'S OBLATENESS ON THE CALCULATION OF THE IMPACT POINT OF BALLISTIC MISSILES[1]

I. INTRODUCTION

The effect of the Earth's oblateness on the calculation of the trajectory of long-range ballistic missiles may be considered in two different aspects, the dynamic effect and the geometric effect. The former comes from the non-Newtonian gravitational force which has a potential (Ref. 1)

$$U = \frac{GM}{R_0}\left[\frac{R_0}{r} + J\frac{R_0}{r^3}\left(\frac{1}{3} - \sin^2 \phi\right) + \ldots\right]$$

where

G = the constant of gravitation

M = the Earth's mass

R_0 = the Earth's equational radius

r = the radius measured from the center of Earth

ϕ = the latitude

J = a constant ($= 1.638 \times 10^{-3}$)

Generally speaking, this dynamic effect is quite small and is, therefore, neglected in most of the existing analyses concerning missile trajectory or

[1]This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under contract No. NASw-6, sponsored by the National Aeronautics and Space Administration.

orbit of space vehicles. The second effect is due to the non-sphericity of the Earth. This effect may be of interest in the calculation of the impact point, the point where the missile trajectory intersects the Earth's surface.

In the following discussions the dynamic effect will be ignored. However, the geometric effect will be taken into account in order to generalize the analysis given by Singer and Wentworth (Ref. 2).

To calculate the impact point, we shall consider the entire trajectory as having two successive stages — namely the ballistic phase and the re-entry phase. The ballistic phase starts from the burnout point and terminates at a hypothetical "re-entry point". In this phase, the missile will move along an elliptical trajectory whose geometrical configuration depends entirely upon the magnitude and direction of the burnout velocity, and the burnout altitude. The aerodynamic drag is neglected during the ballistic flight. However, in the re-entry phase, the performance characteristics of the missile will be entirely different. After the missile re-enters the Earth's atmosphere it will experience tremendous aerodynamic friction because of its extremely high flight speed. As a consequence, aerodynamic heating will appear as a serious problem. Discussion of this point will be excluded in this note; further information may be obtained from Ref. 3. In the analysis of the re-entry phase, the terminal conditions of the ballistic phase may be used as initial conditions. Again, since the time of flight during the re-entry phase is short compared to the time of the ballistic phase, the rotation of the Earth may be neglected. The calculation of the re-entry phase trajectory can be readily performed by applying the analysis of Allen and Eggers, Jr. (Ref. 3).

II. ANALYSIS

If we consider the Earth to be non-spherical and the atmospheric shell similarly oblate, the effect of non-sphericity on the determination of missile impact point may be seen to be of interest only in the ballistic phases. This follows because the oblateness of the atmospheric shell will shift the re-entry point of the rocket when the altitude of the atmospheric shell is fixed.

The elliptical trajectory during the ballistic phase of the missile will be determined if the burnout velocity vector and burnout altitude of the rocket are given. The trajectory may be described by a polar coordinate system designated within the plane of the missile trajectory with origin at the center of the Earth, as shown in Fig. 1. Again, if ρ is the polar distance, the ballistic trajectory is given by the well known equation

$$\rho = \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos \varphi} \tag{1}$$

where

φ = the polar angle measured from the perigee (see Fig. 1)

a = the semimajor axis of the orbit ellipse

ϵ = the eccentricity of the orbit ellipse

$$\epsilon = (1 - \frac{b^2}{a^2})^{1/2}$$

b = the semiminor axis of the orbit ellipse

According to Singer and Wentworth,

$$a = \frac{GM}{\left(\frac{2GM}{R_L} - v_L^2\right)}$$

$$\epsilon = \left[1 - \frac{\left(\frac{2GM}{R_L} - v_L^2\right)R_L v_L^2 \cos^2\theta}{(GM)^2}\right]$$

where v_L is the burnout velocity, R_L is the radial distance from the center of the Earth and the burnout point, G is the gravitational constant, and M is the mass of Earth.

The International Ellipsoid of Earth may be represented also in the terrestrial coordinates

$$(1 - e^2)(r^2 \sin^2\varphi - R_0^2) + r^2 \cos^2\varphi = 0 \tag{2}$$

where

$$e = \frac{\text{equatorial radius - polar radius}}{\text{equatorial radius}}$$

$$\varphi = \pi/2 - \text{latitude}$$

$$R_0 = \text{equatorial radius}$$

Hence, for a given latitude ϕ, the local radius of the Earth may be determined from

$$r = \frac{(1 - e)R_0}{\left[1 - e(2 - e)\sin^2\phi\right]^{1/2}} \tag{3}$$

with $\epsilon \simeq 1/297$. Since ϵ is small, it is possible to write

$$r \simeq (1 - \frac{\bar{e}}{2} \cos^2 \phi) R_0 \tag{4}$$

where $\bar{e} = e(2 - e)$.

If the orientation of the trajectory plane and the orientation of the equatorial plane are different by an angle Φ_0, then the relation between the angle ϕ and the angle of advance of the missile, namely β, may be determined such that (Ref. 4)

$$\cos^2 \phi = \sin^2 \Phi_0 \cos^2 \beta \tag{5}$$

where β is measured from a plane containing the north pole and Earth's center, but perpendicular to the plane of trajectory (see Fig. 2) since it is possible to write

$$\beta = \varphi - \lambda_0$$

where λ_0 is known if the perigee of the "orbit ellipse" is fixed. Thus

$$r = \left[1 - \frac{\bar{e}}{2} \sin^2 \Phi_0 \cos^2 (\varphi - \lambda_0) \right] R_0 \tag{6}$$

The angle φ_E at the re-entry point may be determined by equating (Eq. 1) and (Eq. 6), that is, by solving

$$\frac{h}{R_0} + \left[1 + \frac{\bar{e}}{2} \sin^2\Phi_0 \cos^2(\varphi_E - \lambda_0)\right] = \frac{a}{R_0} \frac{(1 - \epsilon)}{1 + \epsilon \cos\varphi_E} \tag{7}$$

where h is the thickness of the spherical shell.

If the difference of altitude between the burnout point and re-entry point for a spherical atmospheric shell with radius $(R_0 + h)$ is denoted

$$k = R_L - (R_0 + h)$$

Thus, after taking into account the oblateness of the Earth, the difference of altitude between the burnout point and re-entry point, namely k, becomes

$$\bar{k} = k + R_0 \left[\frac{\bar{e}}{2} \sin^2\Phi_0 \cos^2(\varphi_E - \lambda_0)\right]$$

Since in general $\bar{k}$ is small compared to R_L, we can obtain θ_E, (Ref. 5) the angle between the trajectory and radial line (as shown in Fig. 2) at the re-entry point from the relation

$$\theta_E = \frac{\pi}{2} - \theta + \left[\frac{k}{R_L} + \frac{R_0}{R_L}\frac{\bar{e}}{2} \sin^2\Phi_0 \cos^2(\varphi - \lambda_0)\right] \cos^2\theta \left[1 + \tan\theta \right.$$

$$\left. - GM_E(R_L v_L^2 \cos^2\theta)^{-1}\right]$$

Again the velocity at re-entry point v_E and the total time of flight during the ballistic phase t_B can be determined as follows:

$$v_E = v_L - \left[\frac{k}{R_L}\,\frac{R_0}{R_L}\,\frac{\bar{e}}{2}\sin^2\Phi_0\cos^2(\varphi - \lambda_0)\cdot v_L\right]\left[1 + \sin^2\theta\right.$$

$$\left. + \frac{1}{2}\sin 2\theta\left(1 - \frac{G_M}{R_L v_L^2\cos^2\theta}\right)\right]$$

and

$$t_B = t + \left[\frac{k}{R_L} + \frac{R_0}{R_L}\,\frac{\bar{e}}{2}\sin^2\Phi_0\cos^2(\varphi - \lambda_0)\right]\frac{T_e\sqrt{1-\epsilon^2}}{2\pi}\left\{\frac{\epsilon\left[\epsilon + \cos\varphi(R_L)\right]}{\left[1 + \epsilon\cos\varphi(R_L)\right]^2}\right.$$

$$\left. - \frac{1}{1+\epsilon}\sec^2\left[\frac{\varphi(R_L)}{2}\right]\right\}$$

where

$$t = T_e - 2\,\frac{1-\epsilon^2}{2\pi}\left\{-\frac{\sin\varphi(R_L)}{1+\epsilon\cos\varphi(R_L)} + \frac{2}{(1+\epsilon^2)^{1/2}}\tan^{-1}\left[(1-\epsilon^2)\left(\frac{\tan\frac{1}{2}\varphi(R_L)}{1+\epsilon}\right)\right]\right\}T_e$$

with

$$T_e = 2\pi(GM)^{1/2}\,a^{3/2}$$

After θ_E, v_E, and R_E are known, the calculation of the re-entry phase may be readily started. Discussion of the procedure is not considered in this note.

III. REMARKS

It is seen that the effect of oblateness is generally small, especially for small Φ_0. However, according to a rough estimation, the spherical assumption may introduce an error of about 50 km in the calculation of impact point when $a = 1.1\ R_0$, $\epsilon = 0.4$, and $\Phi_0 = \lambda_0 = \pi/2$ are assumed.

Again, since the coordinates are assumed to be fixed in space, in order to determine the angle Φ_0, the local linear velocity at burnout point due to the Earth's rotation should be considered. In order to calculate the correct latitude of the re-entry point the rotation of the Earth must be taken into account. This may easily be calculated since the total time of flight is known. Since no new feature will appear in the present case, discussion along this line is omitted.

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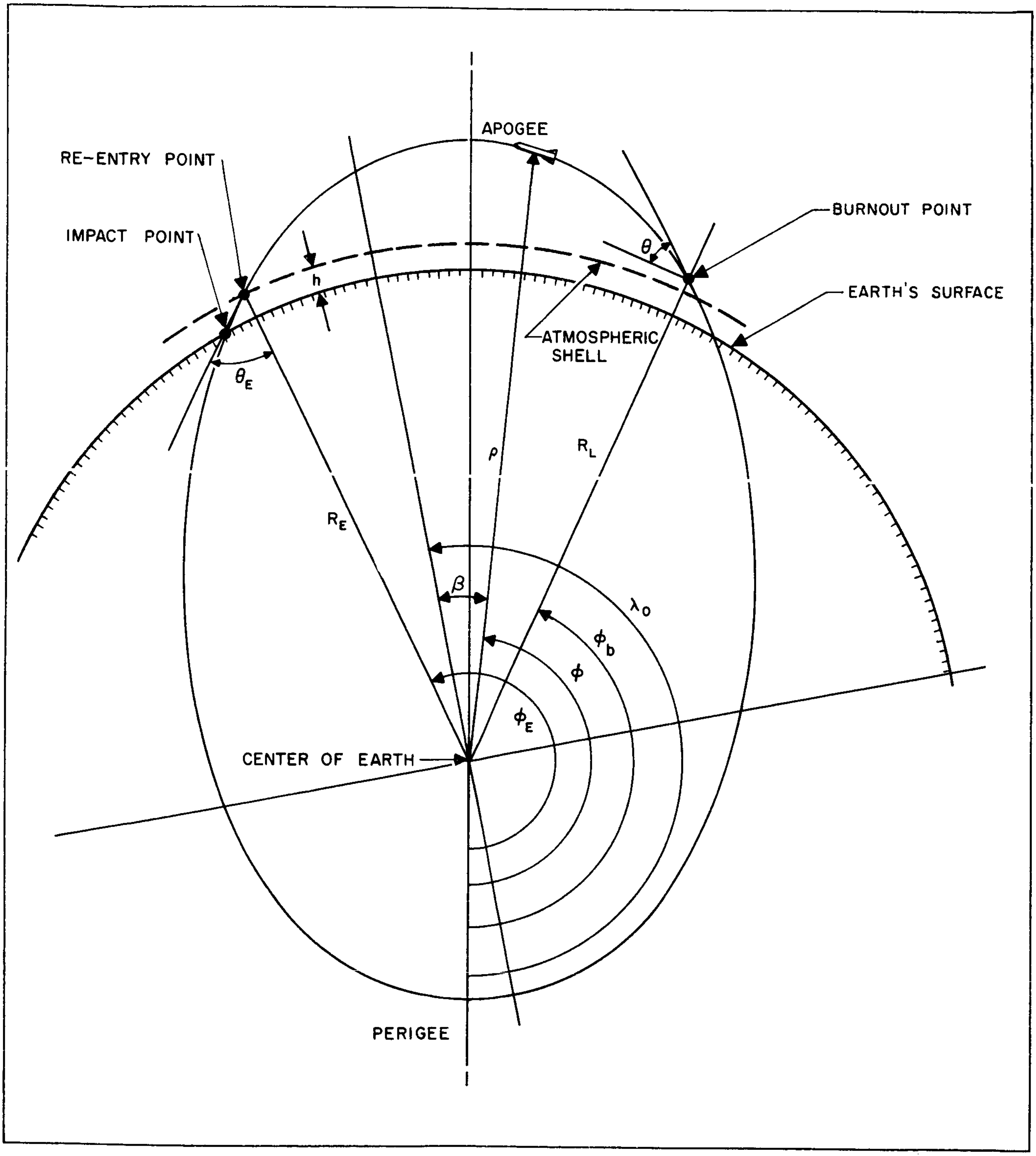

Fig. 1. Elliptical trajectory of the missile

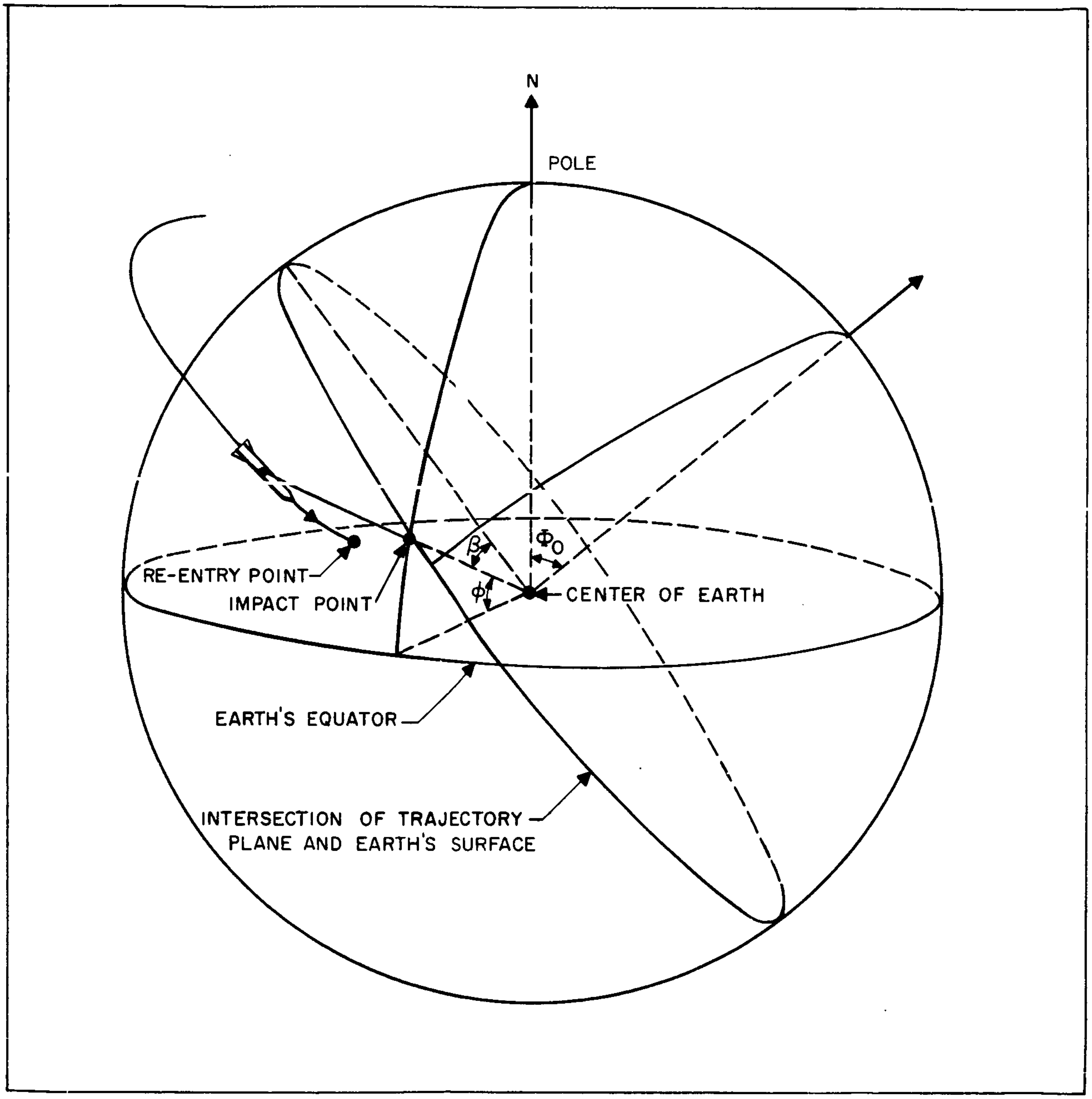

Fig. 2. Geometrical relation between the trajectory plane and the equatorial plane